Appendix C from J. Usinowicz, "Limited Dispersal Drives Clustering and Reduces Coexistence by the Storage Effect" (Am. Nat., vol. 186, no. 5, p. 000)

Nucleation and Exponential Kernels

The theory developed in appendix B for the specific case of cellular automata type dispersal can be generalized to other types of dispersal kernels by finding appropriate expressions for $\overline{N_{\mathrm{I}}(c_i)}$, $\overline{N_{\mathrm{R}}(c_i)}$, and $\overline{N_{\mathrm{M}}(c_i)}$. Here I give versions of equations (B8) through (B10) that have been adapted for exponential kernels (e.g., Gaussian) and the corresponding modifications to equations (B14) through (B16). Then, the calculation of parameters for the nucleation difference equation and c_{c} follows from the rest of appendix B.

In the following sections, I assume that $K(x, y; \Phi)$ is a discretized two-dimensional exponential kernel, standardized such that $\sum_{x\in X}\sum_{y\in Y} K(x, y;\ \mu, \sigma) = 1$. Here μ and σ are stand-ins for parameters defining the shape of the kernel (e.g., the mean and variance in a Gaussian kernel). I assume further that the kernel is symmetric, so that it can be described in terms of a generic distance variable n. The standardization requirement can be written as $\sum_{n\in A} K(n;\ \mu, \sigma) = 1$, where the sum is now over the area A encompassed by the kernel.

Because $\overline{N_{\mathrm{I}}(c_i)}$ and $\overline{N_{\mathrm{R}}(c_i)}$ are both in terms of number of accessible sites, there is little that needs to be changed. The only challenge is to determine alternatives for d_φ and Φ based on kernels that technically extend to infinity in each direction. The simplest solution is to identify a value of n such that $\sum_{n\in N} K(n;\ \mu, \sigma)$ accounts for most of the total propagules produced. To this end, the average dispersal distance works well. For example, in radially symmetric exponential and Gaussian dispersal kernels, the average dispersal distance can be calculated from

$$r_\varphi = \frac{\alpha\Gamma(3/c)}{\Gamma(2/c)}. \tag{C1}$$

Here, Γ is the gamma function, c is a shape parameter ($c = 1$ for exponential and $c = 2$ for Gaussian kernels), and α is the dispersion parameter ($\alpha = 2\sigma$ for Gaussian kernels; Clark 1998). Then d_φ and Φ follow, and $\overline{N_{\mathrm{I}}(c_i)}$ and $\overline{N_{\mathrm{R}}(c_i)}$ are exactly equations (B8) and (B9).

A new expression is required for $\overline{N_{\mathrm{M}}(c_i)}$. In the cellular automata kernel (app. B), $\overline{N(c_i)}$ represented the total number of shared edges between resident and invader, assuming that cluster growth has both radial and linear tendencies. When considering an exponential kernel, a combination of radial and linear growth is still assumed, but now each shared edge is weighted by its distance from reproductive individuals. For radial growth, the concentric perimeters of the cluster (moving from the outer edge inwards) are still equal to $2\pi\sum_{j=0}^{r_\varphi-1}(r_{\mathrm{c}}-j)$. However, the number of propagules reaching each concentric perimeter—the weight of each edge—is a function of the dispersal kernel so that the radial contribution becomes $2\pi\sum_{j=0}^{r_\varphi-1}(r_{\mathrm{c}}-j)\sum_{n=j}^{r_\varphi}K(n)$, where $K(n)$ is the value of the dispersal kernel at distance n. For the linear component of growth, the number of invader-resident edges is again calculated as the size of the neighborhood, minus the invader site at the center and its neighbors. Since the normalized dispersal kernel is equal to 1, this gives $1-\sum_{n=1}^{d_\varphi}K(n)$. Finally, for the quadratic term, exclude the central site from the neighborhood of the kernel so that the term becomes $1-K(0)/L^2$. This leads to

$$\overline{N(c_i)} = \pi\sum_{j=1}^{r_\varphi-1}\left(\sqrt{\frac{c_i}{\pi}}-j\right)\sum_{n=j}^{r_\varphi}K(n) + \left(1-\sum_{n=1}^{d_\varphi}K(n)\right)c_i - \frac{1-K(0)}{l^2}c_1^2. \tag{C2}$$

The shortened expressions for $\overline{N_{\mathrm{I}}(c_i)}$, $\overline{N_{\mathrm{R}}(c_i)}$, and $\overline{N_{\mathrm{M}}(c_i)}$ after dropping quadratic terms correspond to equations (B14) through (B16) with the constants h_{I}, h_{M}, f, and g: $h_{\mathrm{I}} = C_{\mathrm{I}} + r_\varphi[\pi(r_\varphi+1)/2 + r_\varphi\sqrt{\pi p_{\mathrm{m}}}/2]$, $h_{\mathrm{M}} = C_{\mathrm{M}} + \sqrt{\pi p_{\mathrm{m}}}/2\sum_j^{r_\varphi}\sum_{n=j}^{r_\varphi}K(n) + \sum_j^{r_\varphi}\sum_{n=j}^{r_\varphi}jK(n)$, $f = d_\varphi + r_\varphi\sqrt{\pi}/2\sqrt{p_{\mathrm{m}}}$, and $g = (1-\sum_{n=1}^{d_\varphi}K(n)) + \sqrt{\pi}/2\sqrt{p_{\mathrm{m}}}\sum_j^{r_\varphi}\sum_{n=j}^{r_\varphi}jK(n)$. Now the constants C_{I} and C_{M} constrain the y-intercept such that $\overline{N_{\mathrm{I}}(c_1)} = \Phi - 1$ and $\overline{N_{\mathrm{M}}(c_1)} = 1 - K(0)$.